MATH 2050C Lecture 8 (Feb 4)

[Problem Set 4 is posted (and revised), due on Feb 19 (two weeks).]

Recall: $\lim(x_n) = x$ iff

$$\forall \varepsilon > 0, \ \exists K \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K$$

Q: Given $\varepsilon > 0$, how to find such a K?

Two simple but useful tools / tricks:

(i) "Fraction comparison"

$$\frac{\text{smaller}}{\text{Bigger}} \leq \frac{\square}{\square} \leq \frac{\text{Bigger}}{\text{smaller}}$$

(ii) Bernoulli's ineq. $\quad (1+x)^n \geq 1 + nx \quad \forall x \geq -1, \ \forall n \in \mathbb{N}$

(iii) Triangle ineq., Reverse Triangle ineq.

Example 1: Let $b \in (0,1)$ be fixed. Show that

$$\boxed{\lim(b^n) = 0}$$

Pf: Since $b \in (0,1)$, we can write

$$b = \frac{1}{1+a} \quad \text{for some } a > 0$$

By Bernoulli's ineq., since $a > 0 \geq -1$,

$$b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} \qquad \text{(\#)}$$

Let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{a\varepsilon}$ (>0) (##)

by Archimedean Property.

$|b^n| < \varepsilon$.

$|b^n| = \frac{1}{(1+a)^n}$

$\leq \frac{1}{1+na}$ $\qquad K > \frac{1}{a\varepsilon}$

$\leq \frac{1}{1+Ka} \leq \frac{1}{Ka} < \varepsilon$

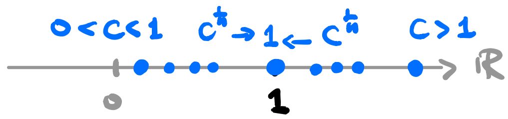
$\forall n \geq K$, we have

$$|b^n - 0| = \frac{1}{(1+a)^n} \overset{(*)}{\leq} \frac{1}{1+na} \leq \frac{1}{1+Ka} \leq \frac{1}{Ka} \overset{(**)}{<} \varepsilon$$

$\blacksquare$

Example 2: Let $c > 0$ be fixed. Show that

$$\boxed{\lim (c^{\frac{1}{n}}) = 1}$$



Pf: Case 1: $c = 1$ then $(c^{\frac{1}{n}}) = (1)$ const seq. (trivial).

Case 2: $c > 1$

Recall $c^{\frac{1}{n}} > 1$ $\forall n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$,

depends on n

$$c^{\frac{1}{n}} = 1 + d_n \quad \text{for some } d_n > 0$$

Raising power n on both sides,

$$c = (c^{\frac{1}{n}})^n = (1 + d_n)^n \geq 1 + n\,d_n$$

Bernoulli's ineq.

Rearrange. $\quad d_n \leq \dfrac{c-1}{n}$ . ... $(*)$

Goal: $c^{\frac{1}{n}} \to 1$.

$|c^{\frac{1}{n}} - 1| = |d_n| = d_n$

Want: $d_n < \varepsilon$

$d_n \leq \frac{c-1}{n} \leq \frac{c-1}{K} < \varepsilon$

$K > \frac{c-1}{\varepsilon} > 0$

Let $\varepsilon > 0$. Take $K \in \mathbb{N}$ s.t. $K \overset{(**)}{>} \dfrac{c-1}{\varepsilon}$ (>0)

$\forall n \geq K$, we have

$$|c^{\frac{1}{n}} - 1| \overset{\text{def}^{\underline{n}} \text{ of } d_n}{=} |d_n| \overset{d_n > 0}{=} d_n \overset{(*)}{\leq} \frac{c-1}{n} \overset{n \geq K}{\leq} \frac{c-1}{K} \overset{(**)}{<} \varepsilon$$

Case 3: $0 < c < 1$.

Recall $0 < c^{\frac{1}{n}} < 1$ $\forall n \in \mathbb{N}$. So, we can write

$$c^{\frac{1}{n}} = \frac{1}{1+h_n} \quad \text{for some } h_n > 0$$

Raise to power n on both sides.

$$c = (c^{\frac{1}{n}})^n = \frac{1}{(1+h_n)^n} \overset{\text{Bernoulli}}{\leq} \frac{1}{1+nh_n} < \frac{1}{nh_n}$$

Rearrange this.

$$h_n < \frac{1}{cn} \quad \text{—— } (\dagger)$$

Let $\varepsilon > 0$ be fixed. Then choose

$$K \in \mathbb{N} \text{ st } K > \frac{1}{c\varepsilon} \; (>0) \quad \text{—— } (\dagger\dagger)$$

Then, $\forall\, n \geq K$, we have

$$|c^{\frac{1}{n}} - 1| \overset{\text{simplify}}{=} \frac{h_n}{1+h_n} < h_n \overset{(\dagger)}{<} \frac{1}{cn} \overset{n \geq K}{\leq} \frac{1}{cK} \overset{(\dagger\dagger)}{<} \varepsilon$$

$$|c^{\frac{1}{n}} - 1| = \left|\frac{1}{1+h_n} - 1\right| = \left|\frac{1-(1+h_n)}{1+h_n}\right| = \left|\frac{-h_n}{1+h_n}\right| = \frac{h_n}{1+h_n} < h_n \overset{?}{<} \varepsilon$$

$$\frac{1}{cn} \leq \frac{1}{cK} < \varepsilon \iff K > \frac{1}{c\varepsilon}.$$

Example 3: $\boxed{\lim (n^{\frac{1}{n}}) = 1}$

Proof: Since $1 \leq n^{\frac{1}{n}}$ $\forall\, n \in \mathbb{N}$, we can write

$$n^{\frac{1}{n}} = 1 + k_n \quad \text{for some } k_n \geq 0$$

$$\Rightarrow \quad n = (1+k_n)^n \quad \left[\text{Trial 1: } \geq 1 + nk_n \Rightarrow k_n \leq \frac{n-1}{n} \overset{?}{<} \varepsilon\right]^{\times} \text{ not enough}$$

$$\geq 1 + \frac{1}{2}n(n-1)k_n^2$$

$$\Rightarrow \quad k_n^2 \leq \frac{\cancel{n-1}}{\frac{1}{2}n\cancel{(n-1)}} = \frac{2}{n} \to 0 \text{ as } n \to \infty$$

So, $0 \le k_n < \sqrt{\frac{2}{n}} \quad \forall n \in \mathbb{N}$.

Let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ s.t. $K > \frac{2}{\varepsilon^2}$. For $n \ge K$,

$$|n^{\frac{1}{n}} - 1| = |k_n| = k_n < \sqrt{\frac{2}{n}} \le \sqrt{\frac{2}{K}} < \varepsilon$$

$\square$